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Convergence of Multipoint Padé-type Approximants

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Let μ be a finite positive Borel measure whose support is a compact subset K of the real line and let I be the convex hull of K. Let r denote a rational function with real coefficients whose poles lie in $\mathbf{C} \setminus I$ and $r(\infty) = 0$. We consider multipoint rational interpolants of the function

$$f(z) = \int \frac{d\mu(x)}{z-x} + r(z),$$

where some poles are fixed and others are left free. We show that if the interpolation points and the fixed poles are chosen conveniently then the sequence of multipoint rational approximants converges geometrically to f in the chordal metric on compact subsets of $\overline{\mathbf{C}} \setminus I$. © 2001 Academic Press

1. INTRODUCTION

Let

$$f(z) = \sum_{m=0}^{\infty} \frac{c_m}{z^m}$$

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be a function which is holomorphic on a neighbourhood of the point $z=\infty$. For each given nonnegative integer n there are polynomials p_n and q_n of degree at most n such that $q_n \not\equiv 0$ and

$$(q_n f - p_n)(z) = \mathcal{O}(1/z^{n+1}), \qquad z \to \infty.$$

The ratio p_n/q_n of any two such polynomials defines a unique rational function π_n which is called the nth (diagonal) Padé approximant of f. It is also possible to define the function π_n as the rational function of order at the most n which has maximal order of contact (within the class of all such functions) with the function f at the point $z=\infty$. Unlike Taylor polynomials, the study of the convergence of the sequence of Padé approximants to f encounters serious difficulties. For instance, there exist entire functions whose sequence of Padé approximants diverges at every point of the complex plane (see, for details, [16]). One of the first results of general character on the convergence of such approximants was obtained by A. A. Markov (see [12], Chapter 3, Theorem 6.1):

Let μ be a finite positive Borel measure whose support, denoted by $S(\mu)$, is a compact subset of the real line $\mathbf{R}$. The so-called Markov function is the function $\hat{\mu}(z)$ defined by

$$\hat{\mu}(z) = \int \frac{d\mu(x)}{z-x}, \qquad z \in \overline{\mathbf{C}} \setminus S(\mu).$$

Let I be the convex hull of $S(\mu)$. Then, the sequence $\{\pi_n\}_{n\in\mathbf{N}}$ of Padé approximants of $\hat{\mu}$ converges uniformly to $\hat{\mu}$ inside (on compact subsets of) the domain $\overline{\mathbf{C}} \setminus I$.

This classical theorem admits several generalizations. It is possible to construct rational approximants interpolating the Markov function along a table of points. In this way, multipoint Padé approximants are obtained and the corresponding Markov theorem for this class of interpolants may be found in [9]. Since the set of singular points of the Markov function is contained in the support of the measure μ we can take advantage of this fact fixing all or part of the poles of the approximant precisely on the set $S(\mu)$. These approximants are commonly called in recent years Padé-type approximants, and Markov-type results involving them have been proved (cf. [6], see also [1] and [3]). Finally, both types of approximation may be combined to give multipoint Padé-type approximants (see definitions below). In this setting we can mention references [4] and [5].

A related problem was posed by A. A. Gonchar in [6]. Let us consider the function

$$f(z) = \int \frac{d\mu(x)}{z-x} + r(z),$$

where r is a rational function whose poles lie in $\mathbf{C} \setminus I$ and $r(\infty) = 0$. Now, f is a meromorphic function on $\mathbf{C} \setminus I$; the poles of f and their order are unknown and should be found by means of the approximants so, rational functions with fixed poles no longer work and it is necessary to use approximants with free or partially free poles. In [6] Gonchar proved the convergence of the sequence of Padé approximants to f locally uniformly in the region obtained from $\mathbf{C} \setminus I$ by deleting the poles of r, under the condition that the absolutely continuous part of the measure μ is positive almost everywhere in $S(\mu)$. In the proof, ratio asymptotics of orthogonal polynomials is strongly used. Later, E. A. Rakhmanov showed that convergence does not hold for arbitrary positive measures μ and general rational function r (cf. [11]); this is due to the possible bad behaviour of the poles of the approximants. Many results in rational approximation (for instance, see [7], Lemma 1, and [8]) point out that the key ingredient to prove convergence is to maintain the poles of the approximants under control. If the coefficients of r are required to be real then all of the poles of the rational approximants, except for a number independent of the order of the approximants, are in I. This fact was used by Rakhmanov to obtain the convergence of the sequence of Padé approximants to the function f without any restriction on the measure μ (see [11]) when the coefficients of r are real.

The aim of this paper is to extend this work of Rakhmanov to the case of multipoint Padé-type approximants.

2. DEFINITIONS AND MAIN RESULTS

As above, let μ be a finite positive Borel measure whose support, denoted by $S(\mu)$, is a compact subset of the real line $\mathbf{R}$ and contains infinitely many points. Otherwise, the Markov function is a rational function. Set $\hat{\mu}(z) = \int (z-x)^{-1} \, d\mu(x)$. Let I be the convex hull of $S(\mu)$. Let r be a rational function with real coefficients whose poles lie in $\mathbf{C} \setminus I$ and $r(\infty) = 0$. The set of poles of r will be denoted by $\mathscr{P}$. Set

$$f(z) = \hat{\mu}(z) + r(z), \qquad z \in \overline{\mathbf{C}} \setminus (S(\mu) \cup \mathscr{P}); \qquad r(z) = \frac{s_d(z)}{t_d(z)},$$

where $\deg s_d \leqslant d-1$, $\deg t_d = d$. We also assume that s_d and t_d have no common factors.

Let $\{L_n\}$, $n \in \mathbf{N}$, be a sequence of monic polynomials whose zeros lie in I. This condition may be replaced by the slightly weaker one that all the limit points of the zeros of L_n are in I. Let us assume that $\deg L_n = k(n) \leqslant n$ and $n - k(n) > 2d$. Let us fix another family of monic polynomials

$$w_n(z) = \prod_{i=1}^{2n} (z - w_{n,i}),$$

whose zeros are contained in a compact set $\mathbf{L} \subset \overline{\mathbf{C}} \setminus (I \cup \mathscr{P})$ and lie symmetrically with respect to the real line, counting multiplicities. In case that for some i, $w_{n,i} = \infty$, the corresponding factor must be omitted. Without loss of generality we may assume that w_n and t_d are positive on $S(\mu)$.

It is easy to verify, keeping in mind the definitions above, that for each $n \in \mathbf{N}$ there exists a unique rational function $\Pi_n(f) = p_n/(q_n L_n^2)$, where p_n and q_n satisfy:

- $\deg q_n \leqslant n - k(n)$, $\deg p_n \leqslant n + k(n) - 1$, and $q_n \not\equiv 0$.
- $q_n L_n^2 f - p_n/w_n \in \mathscr{H}(\mathbf{C} \setminus (S(\mu) \cup \mathscr{P}))$, where $\mathscr{H}(A)$ denotes the set of all holomorphic functions defined on the set $A \subset \overline{\mathbf{C}}$.
- $(q_n L_n^2 f - p_n/w_n)(z) = \mathcal{O}(1/z^{n-k(n)+1})$, $z \to \infty$.

$\Pi_n(f)$ is the *multipoint Padé-type approximant of f* with preassigned poles at the zeros of the polynomial L_n^2, which interpolates the function f at the zeros of the polynomial w_n.

Let ρ_n and ρ be finite Borel measure son $\overline{\mathbf{C}}$. By $\rho_n \xrightarrow{*} \rho$, $n \to \infty$, we denote the weak* convergence of ρ_n to ρ as n tends to infinity. This means that for every continuous function f on $\overline{\mathbf{C}}$

$$\lim_{n \to \infty} \int f(x)\, d\rho_n(x) = \int f(x)\, d\rho(x).$$

For a given polynomial T, we denote by Λ_T the *normalized zero counting measure* of T. That is

$$\Lambda_T = \frac{1}{\deg T} \sum_{\xi:\, T(\xi) = 0} \delta_\xi.$$

The sum is taken over all the zeros of T and δ_ξ denotes the Dirac measure concentrated at ξ.

In the following, for each n, it is considered that $\deg w_n = 2n$, assigning to these polynomials $2n - \deg w_n$ "zeros" at infinity in case that $\deg w_n < 2n$.

It is said that the *sequence of polynomials* $\{w_n\}_{n \in \mathbf{N}}$ *has the measure* v *as its asymptotic zero distribution* if

$$\Lambda_{w_n} \xrightarrow{*} v, \qquad n \to \infty.$$

Let $\{\varphi_n\}_{n \in \mathbf{N}}$ be a sequence of functions defined on a domain D. We will say that the sequence $\{\varphi_n\}_{n \in \mathbf{N}}$ *converges in capacity* to the function φ on compact subsets of D if for each compact subset C of D and for each $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \operatorname{cap}\{z \in C : |\varphi_n(z) - \varphi(z)| > \varepsilon\} = 0,$$

where $\operatorname{cap}(\cdot)$ stands for the logarithmic capacity.

Let K be a compact subset of the real line $\mathbf{R}$ with $\operatorname{cap}(K) > 0$; we will say that K is *regular* if the domain $\overline{\mathbf{C}} \backslash K$ is regular with respect to the Dirichlet problem.

Denote $\Omega = \overline{\mathbf{C}} \backslash S(\mu)$ and let τ be a positive measure supported on Ω. The *Green potential of the measure* τ *in* Ω is the function $G_\Omega(\tau; \cdot)$ defined by

$$G_\Omega(\tau; z) = \int g_\Omega(z; \zeta)\, d\tau(\zeta), \qquad z \in \Omega,$$

where $g_\Omega(z; \zeta)$ is the (generalized) Green function of Ω with singularity at the point ζ.

Let f be a bounded function defined on K. Set $\|f\|_K = \sup\{|f(z)|: z \in K\}$. Finally, let us introduce the main sufficient condition to prove the theorem below. Set $w(z) = \exp(-\int \log|z-t|\, dv(t))$. We will require

$$\limsup_{n \to \infty} \|w^{k(n)} L_n\|_{S(\mu)}^{1/k(n)} \leqslant \exp(-F_w), \tag{1}$$

where F_w is the modified Robin constant for w (see Section 3.2 for details and Section 5 for remarks about this condition). The limit above relates to the behaviour of the zeros of the polynomials L_n on $S(\mu)$.

We are ready for

THEOREM 2.1. *Suppose that the sequence of polynomials* $\{w_n\}_{n \in \mathbf{N}}$ *has* v *as its asymptotic zero distribution. Let* $\operatorname{cap}(S(\mu)) > 0$. *If either* $k(n) = o(n)$ *or* (1) *takes place, then*

1. *For all sufficiently large* n, $\deg q_n = n - k(n)$; *for such* n *the number of poles of* $\Pi_n(f)$ *in* $\overline{\mathbf{C}} \backslash I$ *equals the number of poles of* r; *and the poles of* $\Pi_n(f)$ *in* $\overline{\mathbf{C}} \backslash I$ *tend, as* $n \to \infty$, *to the poles of* r (*in such a way that each pole of* r "*attracts*" *exactly a number of poles equal to its order*).

2. *On each compact subset* K *of* $\overline{\mathbf{C}} \backslash (I \cup \mathscr{P})$, *we have*

$$\limsup_{n \to \infty} \|f - \Pi_n(f)\|_K^{1/2n} \leqslant \|\exp\{-G_\Omega(v; \cdot)\}\|_K.$$

The combination of 1 and 2 indicates that $\Pi_n(f)$ converges to f uniformly on compact subsets of $\overline{\mathbf{C}} \backslash I$ in the chordal metric. Since it is known that $\|\exp\{-G_\Omega(v; \cdot)\}\|_K < 1$, we obtain geometric rate of convergence of the multipoint Padé-type approximants to the function f.

In particular, if $f = \hat{\mu}$, we obtain

COROLLARY 2.1. *Suppose that the sequence of polynomials* $\{w_n\}_{n\in\mathbf{N}}$ *has* v *as its asymptotic zero distribution. Let* $\operatorname{cap}(S(\mu))>0$. *If either* $k(n)=o(n)$ *or* (1) *takes place then, on each compact subset* K *of* $\overline{\mathbf{C}}\setminus I$, *we have*

$$\limsup_{n\to\infty} \|\hat{\mu}-\Pi_n(\hat{\mu})\|_K^{1/2n} \leqslant \|\exp\{-G_\Omega(v;\cdot)\}\|_K.$$

This last result is closely related to those that appear in [3] or [5], though the methods are somewhat different. We will discuss their connection later on.

To conclude this section, we give, for the sake of clarity, a list of the main symbols and notation used in the paper.

μ	A finite positive Borel measure
$S(\mu)$	The support of μ, a compact subset of $\mathbf{R}$
I	The convex hull of $S(\mu)$
$\hat{\mu}(z)$	$\int d\mu(x)/(z-x)$, the Markov function of μ
Ω	$\overline{\mathbf{C}}\setminus S(\mu)$
$r=s_d/t_d$	A rational function with real coefficients
d	Degree of t_d
$\mathscr{P}$	Set of poles of r
$f=\hat{\mu}+r$	A Markov-type meromorphic function
$\{L_n\}_{n\in\mathbf{N}}$	A sequence of monic polynomials whose zeros lie on I
$\mathbf{L}$	A compact subset of $\overline{\mathbf{C}}\setminus(I\cup\mathscr{P})$
$\{w_n\}_{n\in\mathbf{N}}$	A sequence of monic polynomials whose zeros lie on $\mathbf{L}$
$\Pi_n(f)=p_n/(q_nL_n^2)$	The multipoint Padé-type approximant of f
Λ_T:	The normalized zero counting measure of T
$g_\Omega(z,\zeta)$	The Green function of Ω
v	A probability measure supported on $\mathbf{L}$
$G_\Omega(v;\cdot)$	The Green potential of v in Ω
$\operatorname{cap}(\cdot)$	The logarithmic capacity
$P(\tau;z)$	$-\int\log\|z-\zeta\|\,d\tau(\zeta)$, the potential of the measure τ
$w(z)$	$\exp P(v;z)$
F_w	The equilibrium constant associated with w
μ_w	The equilibrium measure associated with w

3. AUXILIARY RESULTS

3.1. *Some Lemmas*

In the sequel, we maintain the notations introduced above. From the definition of multipoint Padé-type approximant it is easy to prove (cf. [4], Lemma 1 or [6], Section 2.3).

LEMMA 3.1. *We have*

$$\int x^j q_n(x) \frac{t_d(x)\, L_n^2(x)}{w_n(x)}\, d\mu(x) = 0, \qquad j = 0, 1, ..., n - k(n) - d - 1 \tag{2}$$

and

$$f(z) = \Pi_n(f)(z) = \frac{w_n(z)}{(t_d q_n L_n^2 h)(z)} \int \frac{(t_d q_n L_n^2 h)(x)}{w_n(x)(z - x)}\, d\mu(x), \tag{3}$$

where h is any polynomial of degree less than or equal to $n - k(n) - d$ and z belongs to $\overline{\mathbf{C}} \setminus (S(\mu) \cup \mathscr{P})$.

From now on, we will represent the polynomials w_n in the following form. For each $n \in \mathbf{N}$, put $w_n = u_n v_n$, where

$$u_n(z) = \prod_{i=1}^{2k(n)+4d} (z - u_{n,i}), \qquad v_n(z) = \prod_{j=1}^{2n - 2k(n) - 2d} (z - v_{n,j}),$$

are such that their zeros lie (in each family) symmetrically with respect to the real line, counting multiplicities. In case that for some i or j, $u_{n,i} = \infty$ or $v_{n,j} = \infty$, the corresponding factor must be omitted. Without loss of generality we may assume that u_n and v_n are positive on $S(\mu)$. We wish to stress that Lemmas 3.2 to 3.4 hold true for any such decomposition of w_n. In the proof of Theorem 2.1, we select u_n and v_n conveniently according to Lemma 3.5.

As a consequence of (2), the polynomial q_n has at least $n - k(n) - d$ changes of sign on I (cf. [15], Section 3.3). Then, q_n can be represented in the form $q_n = q_{n,1} q_{n,2}$, where $\deg q_{n,1} \geqslant n - k(n) - d$ and the zeros $\{x_{n,i}\}$, $i = 1, ..., n'$ of $q_{n,1}$ are simple and belong to I. The polynomial $q_{n,2}$ does not change sign on I; and $\deg q_{n,2} \leqslant d$. Set

$$p_{n,1}(z) = \int \frac{q_{n,1}(z)\, v_n(x) - q_{n,1}(x)\, v_n(z)}{(z - x)\, v_n(x)}\, d\mu_n(x),$$

$$d\mu_n(x) = \frac{t_d q_{n,2} L_n^2}{u_n}(x)\, d\mu(x).$$

LEMMA 3.2. *We have*

$$(t_d q_{n,2} L_n^2)(f - \Pi_n(f))(z) = u_n(z)\left(\hat{\mu}_n(z) - \frac{p_{n,1}}{q_{n,1}}(z)\right), \qquad z \in \overline{\mathbf{C}} \setminus I, \tag{4}$$

$$\frac{p_{n,1}}{q_{n,1}}(z) = \sum_{i=1}^{n'} \frac{\lambda_{n,i}}{z - x_{n,i}},$$
$$\lambda_{n,i} = \int \frac{q_{n,1}(x)\, v_n(x_{n,i})}{q'_{n,1}(x_{n,i})(x - x_{n,i})} \frac{d\mu_n(x)}{v_n(x)}. \tag{6}$$

Proof. From the definition of $p_{n,1}$, we obtain

$$p_{n,1}(z) = q_{n,1}(z)\, \hat{\mu}_n(z) - v_n(z) \int \frac{q_{n,1}(x)}{(z-x)\, v_n(x)}\, d\mu_n(x),$$

or, in an equivalent manner

$$\hat{\mu}_n(z) - \frac{p_{n,1}}{q_{n,1}}(z) = \frac{v_n(z)}{q_{n,1}(z)} \int \frac{q_{n,1}(x)}{(z-x)\, v_n(x)}\, d\mu_n(x). \tag{6}$$

Using Hermite's formula (3) with $h \equiv 1$, we also have

$$\frac{v_n(z)}{q_{n,1}(z)} \int \frac{q_{n,1}(x)}{(z-x)\, v_n(x)}\, d\mu_n(x) = \frac{v_n(z)}{q_{n,1}(z)} \int \frac{(t_d q_n L_n^2)(x)}{(u_n v_n)(x)(z-x)}\, d\mu(x)$$
$$= \frac{(t_d q_{n,2} L_n^2)(z)}{u_n(z)} (f - \Pi_n(f))(z). \tag{7}$$

Now, (6) and (7) together give (4). Furthermore

$$\frac{p_{n,1}}{q_{n,1}}(z) = \hat{\mu}_n(z) - \frac{v_n(z)}{q_{n,1}(z)} \int \frac{(t_d q_n L_n^2)(x)}{(u_n v_n)(x)(z-x)}\, d\mu(x) = o\left(\frac{1}{z}\right),$$

due to the orthogonality relations (2) and taking account of the possible degrees of the polynomials v_n and $q_{n,1}$. Thus $\deg p_{n,1} < \deg q_{n,1}$ and, therefore,

$$\frac{p_{n,1}}{q_{n,1}}(z) = \sum_{i=1}^{n'} \frac{\lambda_{n,i}}{z - x_{n,i}} \qquad \text{with} \quad \lambda_{n,i} = \lim_{z \to x_{n,i}} (z - x_{n,i}) \frac{p_{n,1}}{q_{n,1}}(z).$$

If we now use the integral formula that defines $p_{n,1}$, we obtain (5) and the proof of the lemma is over. ∎

Consider the following linear functional Λ_n. If φ is a function defined on I, then

$$\Lambda_n(\varphi)=\sum_{i=1}^{n'} \lambda_{n,i} \frac{\varphi(x_{n,i})}{v_n(x_{n,i})}.$$

The next lemma is an analog of the Gauss–Jacobi quadrature formula.

LEMMA 3.3. *For every polynomial P with* $\deg P<2n-2k(n)-2d$,

$$\int P(x)\,\frac{d\mu_n(x)}{v_n(x)}=\Lambda_n(P). \tag{8}$$

Proof. Let L, $\deg L<n'$, be the Lagrange polynomial which interpolates a given polynomial P, $\deg P<2n-2k(n)-2d$, at the points $x_{n,i}$, $i=1, ..., n'$. We have $P=L+q_{n,1}T$, where T is a polynomial of degree less than $n-k(n)-d$. Integrating the equation $P=L+q_{n,1}T$, using (2) and (5), we obtain

$$\begin{aligned}\int P(x)\,\frac{d\mu_n(x)}{v_n(x)}&=\sum_{i=1}^{n'} P(x_{n,i})\int \frac{q_{n,1}(x)}{q'_{n,1}(x_{n,i})(x-x_{n,i})}\,\frac{d\mu_n(x)}{v_n(x)}\\&=\Lambda_n(P),\end{aligned}$$

and the proof is complete. ∎

From this last result immediately follows

LEMMA 3.4. *The number of positive coefficients* $\lambda_{n,i}$ *in* (5) *is at least* $n-k(n)-d$.

Proof. Let N be the number of positive $\lambda_{n,i}$ (for a given n). Put $P(z)=\prod^+ (z-x_{n,i})^2$, where $\prod^+$ denotes the product over all indices i for which $\lambda_{n,i}>0$. If $\deg P=2N<2n-2k(n)-2d$, formula (8) is applicable to P, and we obtain a contradiction (the left-hand side of the formula is positive and the right-hand one is nonpositive). Consequently $N>n-k(n)-d$, which proves the lemma. ∎

The next lemma shows that the polynomials u_n and v_n may be chosen so that the sequences inherit the asymptotic zero distribution of $\{w_n\}_{n\in\mathbf{N}}$. Though it may be stated in more general terms we restrict our attention to the case needed for the proof of Theorem 2.1.

LEMMA 3.5. *Suppose that the sequence of polynomials* $\{w_n\}_{n\in\mathbf{N}}$ *with* $\deg w_n = n$, *has the measure* v *as its asymptotic zero distribution and* $S(v) \subset \mathbf{C}$. *Let* $k(n) \in \mathbf{N}$ *be such that* $k(n) \leqslant n$. *If* $\lim_{n\to\infty} k(n) = \infty$ *and* $\lim_{n\to\infty} n - k(n) = \infty$ *then, for each* $n \in \mathbf{N}$, *there exists polynomials* u_n *and* v_n *such that* $w_n = u_n v_n$, $\deg u_n = k(n)$, *and*

$$\Lambda_{u_n} \xrightarrow{*} v, \quad n \to \infty, \qquad \Lambda_{v_n} \xrightarrow{*} v, \quad n \to \infty.$$

Proof. Fix a closed square Q such that the support of v is contained in Q. We may assume, without loss of generality, that all the zeros of the polynomials w_n belong to Q. For each positive integer n, divide Q into $m(n)$ disjoint squares (not necessarily closed): $Q = \bigcup_{j=1}^{m(n)} Q_j^n$. We suppose that $m(n) = o(k(n))$, $m(n) = o(n - k(n))$, and $\lim_{n\to\infty} m(n) = \infty$ verifying

$$\lim_{n\to\infty} \Big(\max_{j=l, \ldots, m(n)} |Q_j^n| \Big) = 0, \qquad \text{where} \quad |A| = \max_{\{a,\, b \in A\}} |a - b|.$$

Now, we construct u_n in the following way. In each square Q_j^n we choose $[k(n)\, \Lambda_{W_n}(Q_j^n)]$ zeros of w_n, where $[\cdot]$ denotes the integer part. These zeros are the zeros of u_n, the rest of them define v_n. These polynomials u_n and v_n do not have the degrees announced in the statement of the lemma. We correct this later. Notice that the difference in degrees is, at most, $m(n)$ for each of them.

The polynomial u_n satisfies

$$\begin{aligned} k(n)\, \Lambda_{w_n}(Q_j^n) - 1 &\leqslant \Upsilon_{u_n}(Q_j^n) \\ &\leqslant k(n)\, \Lambda_{w_n}(Q_j^n), \qquad j = 1, \ldots, m(n), \end{aligned} \tag{9}$$

where Υ_{u_n} stands for the measure which has at each zero of u_n a mass equal to the multiplicity of the zero. From (9) we obtain

$$\Lambda_{w_n}(Q) - \frac{m(n)}{k(n)} \leqslant \frac{\Upsilon_{u_n}(Q)}{k(n)} \leqslant \Lambda_{w_n}(Q).$$

Analogously

$$\Lambda_{w_n}(Q) \leqslant \frac{\Upsilon_{v_n}(Q)}{n - k(n)} \leqslant \Lambda_{w_n}(Q) + \frac{m(n)}{n - k(n)}.$$

Let h be a continuous function on $\overline{\mathbf{C}}$. We may suppose that h is a real, positive function. Denote by M the maximum of h on Q and by M_j^n and m_j^n the maximum and the minimum, respectively, of h on the closure of Q_j^n.

Due to uniform continuity we may suppose that $M_j^n - m_j^n \leqslant \delta_n$, where $\lim_{n\to\infty} \delta_n = 0$. By (9), we have

$$\left| \int_{Q_j^n} h \, d\Lambda_{w_n} - \int_{Q_j^n} h \, \Lambda_{u_n} \right| \leqslant M_j^n \Lambda_{w_n}(Q_j^n) - m_j^n \Lambda_{u_n}(Q_j^n)$$

$$\leqslant (M_j^n - m_j^n) \, \Lambda_{w_n}(Q_j^n) + \frac{m_j^n}{k(n)}.$$

Therefore,

$$\left| \int_Q h \, d\Lambda_{w_n} - \int_Q h \, d\Lambda_{u_n} \right| \leqslant \sum_{j=1}^{m(n)} \left| \int_{Q_j^n} h \, d\Lambda_{w_n} - \int_{Q_j^n} h \, d\Lambda_{u_n} \right|$$

$$\leqslant \sum_{j=1}^{m(n)} \left[(M_j^n - m_j^n) \, \Lambda_{w_n}(W_j^n) + \frac{m_j^n}{k(n)} \right]$$

$$\leqslant \delta_n + M \frac{m(n)}{k(n)},$$

where the right-hand side tends to zero as n tends to infinity. The weak star convergence of Λ_{v_n} is proved in a similar way.

In order to correct the degrees of u_n and v_n according to the statement, we must only transmit from one of these polynomials to the other at most $m(n)$ zeros. Since $m(n) = o(k(n))$ and $m(n) = o(n - k(n))$ the new polynomials preserve the weak star limit of the previous ones. The proof is complete. ▮

The next lemma was proved by Gonchar in [7].

LEMMA 3.6. *Suppose that the sequence* $\{\varphi_n\}$ *of functions defined on the domain* $D \subset \mathbf{C}$ *converges in capacity to a function* φ *on compact subsets of* D. *Then the following assertions hold true*:

1. *If the functions* φ_n, $n \in \mathbf{N}$ *are holomorphic in* D, *then the sequence* $\{\varphi_n\}$ *converges uniformly on compact subsets of* D *and* φ *is holomorphic in* D (*more precisely*, *it is equal to a holomorphic function in* D *except on a set of capacity zero*).

2. *If each of the functions* φ_n *is meromorphic in* D *and has no more than* $k < +\infty$ *poles in this domain, then the limit function* φ *is also meromorphic and has no more than* k *poles in* D.

3. *If each function* φ_n *is meromorphic and has no more than* $k < +\infty$ *poles in* D *and the function* φ *is meromorphic and has exactly* k *poles in* D,

then all φ_n, $n \geqslant N$, also have k poles in D; the poles of φ_n tend to the poles $z_1, ..., z_k$ of φ (taking account of their orders) and the sequence $\{\varphi_n\}$ tends to φ uniformly on compact subsets of the domain $D' = D \setminus \{z_1, ..., z_k\}$.

3.2. *Potential Theory*

Let w be a positive continuous function on $S(\mu)$. Set $g(z) = -\log w(z)$. It is well known (see [14], Sections I.1 and I.3) that among all probability measures σ with support in $S(\mu)$ there exists a unique probability measure μ_w with support in $S(\mu)$, called the extremal or equilibrium measure associated with w, minimizing the weighted energy

$$I_w(\sigma) = \iint \left(\log \frac{1}{|z-t|} + g(z) + g(t) \right) d\sigma(z)\, d\sigma(t).$$

Let $P(\mu_w; z) = -\int \log |z-t|\, d\mu_w(t)$ be the potential of this extremal measure and $S_w \subset S(\mu)$ its support. Under these conditions there exists a constant F_w, called the equilibrium constant or modified Robin constant, such that

$$\begin{aligned} &P(\mu_w; z) + g(z) \geqslant F_w, \qquad z \in S(\mu) \setminus E, \quad \operatorname{cap}(E) = 0, \\ &P(\mu_w; z) + g(z) \leqslant F_w, \qquad z \in S_w. \end{aligned} \tag{10}$$

Due to (10), μ_w is also called the equilibrium measure in the presence of the external field g. The constant F_w is determined by

$$F_w = I_w(\mu_w) - \int g(t)\, d\mu_w(t).$$

If $\{w_n\}_{n \in \mathbf{N}}$ has asymptotic zero distribution v, it is easy to see that $(w_n)^{-1/\deg w_n}$ uniformly converges to $e^{P(v;\,\cdot)}$ on $S(\mu)$, where $P(v; \cdot)$ is the potential of the probability measure v. If we take $g(z) = -P(v; \cdot)$, since the support of v is contained in $\mathbf{L} \subset \overline{\mathbf{C}} \setminus (I \cup \mathscr{P})$, it is well known that μ_w is the balayage of v onto $S(\mu)$ and S_w coincides with $S(\mu)$ minus a set of capacity zero (for instance, see [14], Chapter IV, Theorem 1.10). Therefore,

$$P(\mu_w; z) - P(v; z) = F_w, \qquad z \in S(\mu) \setminus E, \quad \operatorname{cap}(E) = 0. \tag{11}$$

It is also known (see Theorem 5.1, Chapter II, in [14]) that

$$G_\Omega(v; z) = F_w - P(\mu_w; z) + P(v; z), \qquad z \in \Omega = \overline{\mathbf{C}} \setminus S(\mu). \tag{12}$$

Recall that $G_\Omega(v; \cdot)$ is the Green potential of the measure v in Ω. The next lemma tells us that Green potentials behave properly with respect to an increasing union of domains.

LEMMA 3.7. *Let* $\{K_n\}_{n\in\mathbf{N}}$ *be a sequence of compact sets contained in* $\mathbf{R}$ *such that* $K_{n+1}\subset K_n$ *for each* $n\in\mathbf{N}$ *and* $\operatorname{cap}(\bigcap_{n=1}^{\infty} K_n)>0$. *Let* ν *be a positive measure with compact support in* $\mathbf{C}\backslash K_1$. *Then*

$$\lim_{n\to\infty} e^{-G_{D_n}(\nu;\, z)} = e^{-G_D(\nu;\, z)},$$

uniformly on compact subsets of D, *where* $D_n=\mathbf{C}\backslash K_n$ *and* $D=\bigcup_{n=1}^{\infty} D_n$.

Proof. For each $n\in\mathbf{N}$, denote $F_{w,n}$ and $\mu_{w,n}$ the Robin constant and the equilibrium measure, respectively, associated with $w=e^{P(\nu;\,\cdot)}$; where we consider the function w restricted to the set K_n. And let F_w and μ_w be the Robin constant and the equilibrium measure, respectively, associated with the same weight w restricted to the set $\bigcap_{n=1}^{\infty} K_n$. In the situation of this lemma, it is known (cf. Theorems 6.2 and 6.5, Chapter I, in [14]) that

$$\lim_{n\to\infty} F_{w,n}=F_w \quad\text{and}\quad \mu_{w,n}\xrightarrow{*}\mu_w, \qquad n\to\infty.$$

Therefore, if we denote $F_{w,n}-P(\mu_{w,n};\,\cdot)$ by h_n, it is obtained that the functions h_n converge to $F_w-P(\mu_w;\,\cdot)$ uniformly on compact subsets of D. Hence,

$$\lim_{n\to\infty} e^{-h_n(z)} = e^{-F_w+P(\mu_w;\, z)},$$

uniformly on compact subsets of D, since the functions h_n are uniformly bounded on such subsets.

On the other hand, the function $-P(\nu;\,\cdot)$ is a subharmonic function on D (in fact, on all $\mathbf{C}$) so, for any compact subset C of D, $-P(\nu;\,\cdot)$ attains its maximum: say M_C. Then, $e^{-P(\nu;\, z)}\in[0, e^{M_C}]$ $\forall z\in C$, which implies, using (12), that

$$\begin{aligned}\lim_{n\to\infty} e^{-G_{D_n}(\nu;\, z)} &= \lim_{n\to\infty} e^{-P(\nu;\, z)}e^{-h_n(z)}\\ &= e^{-P(\nu;\, z)}e^{-F_w+P(\mu_w;\, z)} = e^{-G_D(\nu;\, z)},\end{aligned}$$

uniformly on C. ▮

4. PROOF OF THE THEOREM

In the sequel, without loss of generality, we may assume that $\mathbf{L}$ is a compact subset of $\mathbf{C}\backslash I$. The reduction to this case may be achieved by means of a Möbius transformation of the variable in the initial problem, which transforms $S(\mu)$ into another compact subset of $\mathbf{R}$ and $\mathbf{L}\subset\overline{\mathbf{C}}\backslash I$ into a compact subset contained in $\mathbf{C}\setminus\tilde{I}$, where $\tilde{I}$ is the image of I by the Möbius transformation. This assumption implies, in particular, that for each n the degree of w_n is really $2n$, and liberates our arguments from the special treatment which otherwise we would have to give to neighbourhoods of infinity.

The proof is divided into three parts. In the first we obtain a general estimate of the size of $f-\Pi_n(f)$ on compact subsets of $\mathbf{C}\setminus I$. Here we must deal with two problems; namely, the poles of $\Pi_n(f)$ (zeros of $q_{n,2}$) which may lie in $\mathbf{C}\setminus I$ (and in fact some do), and the zeros of $q_{n,1}$ which have negative coefficients $\lambda_{n,i}$. Fortunately the number of points with these undesired properties does not depend on n as n tends to infinity. To handle these problems we follow arguments employed in the proof of the Theorem in [10]. The general estimate obtained in the first part allows us to give proper bounds in part two using techniques from potential theory. The bounds of part two provide convergence in capacity of $\Pi_n(f)$ to f on compact subsets of $\mathbf{C}\setminus I$ and with the aid of Gonchar's Lemma we conclude the proof in part three.

1. Let $w_n=u_nv_n$ be any decomposition of w_n as defined in the beginning of Section 3.1. Let $\rho_n(z)=\prod^-(z-x_{n,i})^2$, where $\prod^-$ denotes the product over the indices i for which $\lambda_{n,i}<0$. From Lemma 3.4 it follows that $\deg \rho_n \leqslant 2d$. Let us consider the functions

$$\Phi_n(z)=\frac{\rho_n q_{n,2} t_d L_n^2}{u_n}(z)(f-\Pi_n(f))(z), \qquad z\in\overline{\mathbf{C}}\setminus I.$$

For every $z\in\overline{\mathbf{C}}\setminus I$, from (4) and (5), we obtain

$$\begin{aligned}\Phi_n(z)&=\rho_n(z)\left[\hat{\mu}_n(z)-\frac{p_{n,1}}{q_{n,1}}(z)\right]\\&=\rho_n(z)\left[\hat{\mu}_n(z)-\Lambda_n\left(\frac{v_n(x)}{z-x}\right)\right].\end{aligned}$$

Denote

$$K_n(x;z)=\frac{\rho_n(z)-\rho_n(x)}{(z-x)\,\rho_n(z)};$$

then, K_n is a polynomial (in x) of degree less than $2d$. We have

$$\frac{1}{z-x}-K_n(x;z)=\frac{\rho_n(x)}{\rho_n(z)}\,\frac{1}{z-x}.$$

From the preceding representation of Φ_n and Lemma 3.3, we obtain (recall that $n-k(n)>2d$)

$$\begin{aligned}\Phi_n(z)&=\rho_n(z)\int\left(\frac{1}{z-x}-K_n(x;z)\right)d\mu_n(x)\\&\quad-\rho_n(z)\,\Lambda_n\left(\frac{v_n(x)}{z-x}-v_n(x)\,K_n(x;z)\right)\\&=\int\rho_n(x)\,\frac{d\mu_n(x)}{z-x}-\Lambda_n\left(\rho_n(x)\,\frac{v_n(x)}{z-x}\right),\qquad z\in\overline{\mathbf{C}}\backslash I.\end{aligned}$$

Let K be an arbitrary compact subset of $\overline{\mathbf{C}}\backslash I$. On one hand

$$\left|\int\rho_n(x)\,\frac{d\mu_n(x)}{z-x}\right|\leqslant M(K)\left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)}\|q_{n,2}t_d\|_{S(\mu)},\qquad z\in K,\tag{13}$$

where $M(K)$ denotes a constant which may depend on K but not on n. In the following, in each appearance, $M(K)$ may denote a different constant with the same characteristics.

On the other hand, using Lemma 3.3, we obtain (notice that $\rho_n(x_{n,i})=0$ whenever $\lambda_{n,i}<0$)

$$\begin{aligned}\left|\Lambda_n\left(\rho_n(x)\,\frac{v_n(x)}{z-x}\right)\right|&=\left|\sum_{i=1}^{n'}\lambda_{n,i}\,\frac{\rho_n(x_{n,i})}{z-x_{n,i}}\right|\\&\leqslant\sum_{i=1}^{n'}\lambda_{n,i}\,\frac{\rho_n(x_{n,i})}{|z-x_{n,i}|}\\&\leqslant M(K)\sum_{i=1}^{n'}\lambda_{n,i}\rho_n(x_{n,i})\\&=M(K)\int\rho_n(x)\,d\mu_n(x)\\&\leqslant M(K)\left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)}\|q_{n,2}t_d\|_{S(\mu)},\qquad z\in K.\end{aligned}\tag{14}$$

Therefore, by the use of (13) and (14), we obtain

$$|\Phi_n(z)|\leqslant M(K)\left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)}\|q_{n,2}t_d\|_{S(\mu)},\qquad z\in K,\tag{15}$$

where K is an arbitrary compact subset of $\overline{\mathbf{C}}\backslash I$.

2. Let $\{K_m\}_{m\in\mathbf{N}}$ be a sequence of regular compact sets contained in I such that $K_{m+1}\subset K_m$ and $S(\mu)=\bigcap_{m=1}^{\infty} K_m$. We may assume that each K_m is a finite union of intervals. Let us denote by $\mu_{w,m}$ and $F_{w,m}$ the equilibrium measure on the set K_m and the modified Robin constant, respectively, associated with the external field $w(z)=\exp P(v;z)$. Set $\Omega_m=\overline{\mathbf{C}}\setminus K_m$ and let $G_{\Omega_m}(v;\zeta)$ be the corresponding Green potential. Let m be an arbitrary natural number but fixed. In view of (2), $q_{n,1}$, $n\in\mathbf{N}$, has, at most, $d+1$ simple zeros in each connected component of $I\setminus S(\mu)$. In this way, $q_{n,1}$ may be represented in the form $q_{n,1}=\tilde{q}_{n,1}h_n$, where $\deg h_n=a_n\leqslant A(m)$, $n\geqslant N$, and $\tilde{q}_{n,1}$ has all its zeros in K_m.

Now, let us consider the functions

$$H_n(z)=\frac{\Phi_n(z)\,h_n(z)}{\|u_n^{-1}L_n^2\|_{S(\mu)}\,\|q_{n,2}t_d\|_{S(\mu)}\,v_n(z)\,\rho_n(z)},\qquad z\in\overline{\mathbf{C}}\setminus K_m.$$

From the definition of multipoint Padé-type approximant it is easily seen that

$$H_n\in\mathscr{H}(\overline{\mathbf{C}}\setminus K_m)\quad\text{and}\quad H_n=\mathcal{O}\left(\frac{1}{z^{2n-2k(n)-2d-a_n+1}}\right),\qquad z\to\infty. \tag{16}$$

We also know, due to (15), that

$$|H_n(z)\,v_n(z)|\leqslant M(K),\qquad z\in K, \tag{17}$$

where K is any given compact subset of $\mathbf{C}\setminus I$, since h_n has all its zeros in I and its degree is bounded by $A(m)$.

Notice that

$$\begin{aligned}&h_n(z)[f(z)-\Pi_n(f)(z)]\\&\quad=\left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)}\frac{u_n(z)}{L_n^2(z)}\,H_n(z)\,v_n(z)\,\frac{\|q_{n,2}t_d\|_{S(\mu)}}{q_{n,2}(z)\,t_d(z)},\end{aligned} \tag{18}$$

where z belongs to $\mathbf{C}\setminus I$. Our next goal is to prove that the sequence

$$\left\{\left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)}\frac{u_n(z)}{L_n^2(z)}\,H_n(z)\,v_n(z)\right\}_{n\in\mathbf{N}} \tag{19}$$

converges to zero with a geometric rate uniformly on compact subsets of $\mathbf{C}\setminus I$. To this end, we estimate separately the factors

$$H_n(z)\,v_n(z)\qquad\text{and}\qquad\left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)}\frac{u_n(z)}{L_n^2(z)}.$$

We start with the factors of the first kind. Let K be an arbitrary compact subset of $\mathbf{C}\backslash I$. Set $\gamma_\varepsilon = \{\zeta \in \mathbf{C} : \exp G_{\Omega_m}(v; \zeta) = 1+\varepsilon\}$, where ε is a positive constant sufficiently small so that K and $\mathbf{L}$ lie in the unbounded component of $\mathbf{C}\backslash\gamma_\varepsilon$. In view of (16), for each n, the function

$$|H_n(z)|\,[\exp(F_{w,m} - P(\mu_{w,m}; z))]^{2n-2k(n)-2d-a_n}$$

is subharmonic in $\overline{\mathbf{C}}\backslash K_m$. Taking (17) into account, we obtain that

$$\begin{aligned}|H_n(z)\,v_n(z)|\,[\exp(F_{w,m} - P(\mu_{w,m}; z))]^{2n-2k(n)-2d-a_n}\\ \leqslant M(\gamma_\varepsilon)[\exp(F_{w,m} - P(\mu_{w,m}; z))]^{2n-2k(n)-2d-a_n}\\ \leqslant M(\gamma_\varepsilon)[\exp(F_{w,m} - P(\mu_{w,m}; z))]^{2n-2k(n)-4d}, \qquad z\in\gamma_\varepsilon,\end{aligned}$$

where $M(\gamma_\varepsilon)$ has the same characteristics as $M(K)$. Or equivalently, using (12)

$$\begin{aligned}|H_n(z)|\,[\exp(F_{w,m} - P(\mu_{w,m}; z))]^{2n-2k(n)-4d-a_n}\\ \leqslant \frac{M(\gamma_\varepsilon)}{|v_n(z)|}\,\frac{[\exp(G_{\Omega_m}(v; z))]^{2n-2k(n)-4d}}{[\exp(P(v; z))]^{2n-2k(n)-4d}}, \qquad z\in\gamma_\varepsilon.\end{aligned} \tag{20}$$

Now, we suppose that $\lim_{n\to\infty} n-k(n)=\infty$ and $\lim_{n\to\infty} k(n)=\infty$ (the bounded cases are easier and are considered at the end of this section). We choose u_n and v_n, $\deg v_n = 2n-2k(n)-4d$, with the additional property of having v as their asymptotic zero distribution as in Lemma 3.5. Since the degree of all the polynomials involved is even this choice may be done in such a way that the zeros of u_n and v_n lie symmetrically with respect to the real line. From the fact that v is the asymptotic zero distribution of $\{v_n\}_{n\in\mathbf{N}}$, we obtain

$$\lim_{n\to\infty} |v_n(z)|^{1/\deg v_n} = e^{-P(v;z)}, \tag{21}$$

uniformly on compact subsets of $\mathbf{C}\backslash\mathbf{L}$, and using the Principle of Descent (see [14], Chapter 1, Theorem 6.8), we have that

$$\limsup_{n\to\infty} |v_n(z)|^{1/\deg v_n} \leqslant e^{-P(v;z)}, \tag{22}$$

uniformly on compact subsets of $\mathbf{C}$. Now, for sufficiently large $n\in\mathbf{N}$, (20) and (21) together give

$$\begin{aligned}|H_n(z)|\,[\exp(F_{w,m} - P(\mu_{w,m}; z))]^{2n-2k(n)-4d-a_n}\\ \leqslant M(\gamma_\varepsilon)\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2n-2k(n)-4d}, \qquad z\in\gamma_\varepsilon.\end{aligned}$$

It follows from the Maximum Principle for subharmonic functions that the same inequality holds for any z in K. Hence, using (22), we obtain that

$$\begin{aligned}|H_n(z)\, v_n(z)| &= |H_n(z)|\, [\exp(F_{w,m} - P(\mu_{w,m}; z))]^{2n-2k(n)-4d-a_n} \\ &\quad \times |v_n(z)|\, [\exp(-F_{w,m} + P(\mu_{w,m}; z))]^{2n-2k(n)-4d-a_n} \\ &\leqslant M(\gamma_\varepsilon)\, |v_n(z)| \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2n-2k(n)-4d} \\ &\quad \times [\exp(-G_{\Omega_m}(v; z)) \exp(P(v; z))]^{2n-2k(n)-4d} \\ &\leqslant M(\gamma_\varepsilon) \left(\frac{(1+\varepsilon)^2}{1-\varepsilon}\right)^{2n-2k(n)-4d} \\ &\quad \times [\exp(-G_{\Omega_m}(v; z))]^{2n-2k(n)-4d}, \qquad z \in K; \end{aligned} \tag{23}$$

for sufficiently large $n \in \mathbf{N}$.

On the other hand, (recall that $w(z) = \exp(P(v; z))$)

$$\begin{aligned}\left\| \frac{L_n^2}{u_n} \right\|_{S(\mu)} &= \left\| \frac{L_n^2 w^{2k(n)+4d}}{u_n w^{2k(n)+4d}} \right\|_{S(\mu)} \\ &\leqslant C \frac{\| L_n^2 w^{2k(n)} \|_{S(\mu)}}{\min_{\zeta \in S(\mu)} \{ u_n(\zeta)\, w^{2k(n)+4d}(\zeta) \}},\end{aligned}$$

where C is an absolute constant which may be different in each appearance.

Therefore, taking account of (1) and (21) relative to u_n, we have that

$$\left\| \frac{L_n^2}{u_n} \right\|_{S(\mu)} \leqslant C \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2k(n)} [\exp(-F_w)]^{2k(n)+4d}, \tag{24}$$

for sufficiently large $n \in \mathbf{N}$.

Condition (1) implies that (see [14], Chapter III, Theorem 4.2)

$$\Lambda_{L_n} \overset{*}{\to} \mu_w, \qquad n \to \infty,$$

which in turn implies that

$$\lim_{n \to \infty} |L_n(z)|^{1/\deg L_n} = e^{-P(\mu_w; z)},$$

uniformly on compact subsets of $\mathbf{C}\backslash S(\mu)$. Using this fact and (22) with u_n instead of v_n, we have that

$$\frac{|u_n(z)|}{|L_n^2(z)|} \leqslant M(K) \frac{|u_n(z)|\,[\exp(P(v;z))]^{2k(n)+4d}}{|L_n^2(z)|\,[\exp(P(\mu_w;z))]^{2k(n)}} \times [\exp(P(\mu_w;z)-P(v;z))]^{2k(n)+4d}$$
$$\leqslant M(K)\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2k(n)} [\exp(P(\mu_w;z)-P(v;z))]^{2k(n)+4d}, \quad z\in K, \tag{25}$$

for sufficiently large $n\in\mathbf{N}$. From (24) and (25), we obtain

$$\left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)} \frac{|u_n(z)|}{|L_n^2(z)|} \leqslant M(K)\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{4k(n)} [\exp(-G_\Omega(v;z))]^{2k(n)+4d}$$
$$\leqslant M(K)\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{4k(n)} [\exp(-G_{\Omega_m}(v;z))]^{2k(n)+4d}, \qquad z\in K, \tag{26}$$

for sufficiently large $n\in\mathbf{N}$.

Using (23) and (26), taking limits, and making ε tend to zero it follows that

$$\limsup_{n\to\infty} \left\| \left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)} \frac{u_n(z)}{L_n^2(z)} H_n(z)\, v_n(z)\right\|_K^{1/2n} \leqslant \|\exp(-G_{\Omega_m}(v;\cdot))\|_K, \tag{27}$$

for each compact subset K of $\mathbf{C}\backslash I$. From here it immediately follows that the sequence (19) converges uniformly to zero, with geometric rate, on each compact subset of $\mathbf{C}\backslash I$.

If $n-k(n)\leqslant B$, $B\in\mathbf{R}$, for all $n\in\mathbf{N}$, it is obvious that $\{u_n\}_{n\in\mathbf{N}}$ has v as its asymptotic zero distribution. Then, we attain (27) by the use of (17) and (26). Finally, we obtain (27) in the general case (regarding the size of $n-k(n)$) passing to subsequences.

3. Following standard arguments, from (18) and (27), it is not difficult to prove convergence in capacity of the sequence $\Pi_n(f)$ to f on compact subsets of $\mathbf{C}\backslash I$ (cf. proof of Theorem 1 in [11]). The poles of the function $\Pi_n(f)$ in $\mathbf{C}\backslash I$ are the zeros of $q_{n,2}$ and their number does not

exceed d. The number of poles of f in $\mathbf{C}\setminus I$ is exactly equal to d, and part 1 of the statement of Theorem 2.1 now follows from Lemma 3.6. Therefore, on every compact set $K\subset\mathbf{C}\setminus(I\cup\mathcal{P})$ the function $q_{n,2}t_d$ can be uniformly bounded below for sufficiently large n. Thus, from (18) and (27), we have that

$$\limsup_{n\to\infty}\|f-\Pi_n(f)\|_K^{1/2n}\leqslant\|\exp(-G_{\Omega_m}(v;\cdot))\|_K, \tag{28}$$

where K is any compact subset of $\mathbf{C}\setminus(I\cup\mathcal{P})$ and m is an arbitrary natural number. Therefore, with the aid of Lemma 3.7, we obtain part 2 of the statement of Theorem 2.1 for compact subsets of $\mathbf{C}\setminus(I\cup\mathcal{P})$. Since $f-\Pi_n(f)$ is holomorphic in a neighbourhood of $z=\infty$ and takes the value 0 there, we have, by the Maximum Principle, that (28) holds true for arbitrary compact subsets in $\overline{\mathbf{C}}\setminus(I\cup\mathcal{P})$, which completes the proof if (1) takes place and $\lim_{n\to\infty}k(n)=\infty$.

Now, we assume that $k(n)=o(n)$. Let K be an arbitrary compact set of $\mathbf{C}\setminus I$. Let $\alpha=\min_{z\in K,\,\zeta\in I}|z-\zeta|$ and $\beta=\max_{z\in S(\mu)\,\zeta\in I}|z-\zeta|$. If $z\in K$ and we use (21) and (22) with u_n instead of v_n, we have that

$$\begin{aligned}\frac{|u_n(z)|}{|L_n^2(z)|}\left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)}&\leqslant\frac{\|L_n^2\|_{S(\mu)}}{|L_n^2(z)|}\frac{|u_n(z)|}{\min_{\zeta\in S(\mu)}|u_n(\zeta)|}\\&\leqslant\left(\frac{\beta}{\alpha}\right)^{2k(n)}\left(\frac{\|\exp(-P(v;\;\cdot\;))\|_K+\varepsilon}{\min_{\zeta\in S(\mu)}|\exp(-P(v;\zeta))|-\varepsilon}\right)^{2k(n)+4d},\end{aligned}$$

for all sufficiently large $n\in\mathbf{N}$. Therefore, if $k(n)=o(n)$, we obtain

$$\limsup_{n\to\infty}\left\|\frac{L_n^2}{u_n}\right\|_{S(\mu)}^{1/2n}\left\|\frac{u_n(z)}{L_n^2(z)}\right\|_K^{1/2n}\leqslant 1,$$

where K is any compact subset of $\mathbf{C}\setminus I$. Now, the proof is analogous to the previous one using this last result, instead of (26).

Finally, if only condition (1) takes place we attain the result passing to subsequences.

5. REMARKS

As we said above, condition (1) implies (assuming $\lim_{n\to\infty}k(n)=\infty$) that

$$\Lambda_{L_n}\overset{*}{\to}\mu_w,\qquad n\to\infty.$$

If the set $S(\mu)$ is regular with respect to the Dirichlet problem in Ω both conditions are equivalent (see Lemmas 3 and 4 in [5]) but, in general, the reciprocal statement is not true. To see this, let us consider the set $[-2, 2] \cup \{3\}$ and the Chebyshev polynomials $T_n(z) = z^n + \cdots$ for $[-2, 2]$. Let μ be the measure on $[-2, 2] \cup \{3\}$ which restricted to $[-2, 2]$ equals the Lebesgue measure and has mass 1 at $z = 3$. We take $u_n \equiv v_n \equiv 1$, for all $n \in \mathbf{N}$. The Chebyshev polynomials T_n verify

$$\limsup_{n \to \infty} \|T_n\|_{[-2,2]}^{1/n} \leqslant \operatorname{cap}([-2, 2]) = 1. \tag{29}$$

This fact is equivalent (see [2], Theorem 1) to

$$\Lambda_{T_n} \xrightarrow{*} \mu_w, \qquad n \to \infty, \tag{30}$$

where $d\mu_w$ is the equilibrium measure of $[-2, 2]$, which is the same measure that the equilibrium measure of $[-2, 2] \cup \{3\}$, since both sets differ in a set of capacity zero. In turn, any one of the conditions (29) and (30) is equivalent to

$$\lim_{n \to \infty} |T_n(z)|^{1/n} = \exp(g_{\mathbf{C} \setminus [-2,2]}(z; \infty)),$$

uniformly on compact subsets of $\mathbf{C} \setminus [-2, 2]$. In particular

$$\lim_{n \to \infty} |T_n(3)|^{1/n} = \exp(g_{\mathbf{C} \setminus [-2,2]}(3; \infty)) > 1. \tag{31}$$

On the other hand, condition (1), for the polynomials T_n, reads

$$\limsup_{n \to \infty} \|T_n\|_{[-2,2] \cup \{3\}}^{1/n} \leqslant \operatorname{cap}([-2, 2] \cup \{3\}) = 1,$$

which contradicts (31). However, it is easy, in this case, to construct a family of polynomials that satisfies (1). It is sufficient to take $L_n(z) = T_n(z)(z-3)$. In general, there always exist families of polynomials verifying (1), for instance, Chebyshev and Fekete (weighted) polynomials (see [14], Chapter 3).

Regarding the relationship between Corollary 2.1 and Theorem 2 in [5], neither of the results is contained in the other. In contrast with [5], we do not require that either $S(\mu)$ be regular or that the measure μ be regular (see [5] for definition). However, our condition on the polynomials L_n is stronger and we do not obtain the exact rate of convergence.

If, in Corollary 2.1, the polynomials w_n are taken to be equal to 1, we may compare this result and Theorem 2′ in [3]. In case that $S(\mu)$ is a regular compact set, both results are equivalent; otherwise, our requirement (1) is stronger than the one that appears in [3], but we let $\lim_{n\to\infty} k(n)/n$ be equal to 1, which is not allowed in [3]. The case $\lim_{n\to\infty} k(n)/n = 1$ corresponds to the case when "almost" all the poles of $\Pi_n(\hat{\mu})$ are fixed. In this situation the construction of the Padé-type approximants has the least computational cost since the zeros of L_n are given.

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